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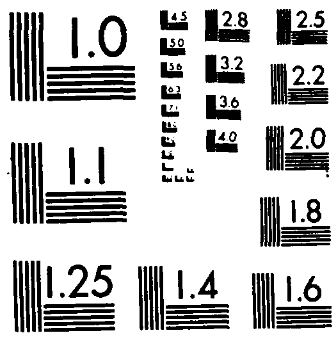


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Define $Y_0$ to be a geometric convolution of X if $Y_0$ is the sum of $N_0$ i.i.d. random variables distributed as X, where $N_0$ is geometrically distributed and independent of X. It is known that if X is non-negative with finite second moment then as $p \to 0$, $Y_0/EY_0$ converges in distribution to an exponential distribution with mean 1. We derive an upper bound for $d(Y_0)$, the distance between $Y_0$ and an exponential with mean $Y_0$, namely for $0 < p \leq 1/2$, $d(Y_0) \leq cp$ where $c = EX^2/(EX)^2$. This bound is asymptotically $(p \to 0)$ tight. Also derived is a bound for $d(Y_0+Z)$ where $Z \geq 0$ is independent of $Y_0$.


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# Error Bounds for Exponential Approximations of Geometric Convolutions

By

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## Abstract

Define $Y_0$ to be a geometric convolution of $X$ if $Y_0$ is the sum of $N_0$ i.i.d. random variables distributed as $X$, where $N_0$ is geometrically distributed and independent of $X$. It is known that if $X$ is non-negative with finite second moment then as $p \to 0$, $Y_0/EY_0$ converges in distribution to an exponential distribution with mean 1. We derive an upper bound for $d(Y_0)$, the distance between $Y_0$ and an exponential with mean $Y_0$, namely for $0 < p \leq 1/2$, $d(Y_0) \leq cp$ where $c = EX^2/(EX)^2$. This bound is asymptotically $(p \to 0)$ tight. Also derived is a bound for $d(Y_0+Z)$ where $Z \geq 0$ is independent of $Y_0$.

1. Introduction. If $\{X_i, i \geq 1\}$ is an i.i.d. sequence and $N_0$ is geometrically distributed $(\Pr(N_0=k) = q^k p,\ k=0,1,2,\ldots)$ and independent of $\{X_i\}$, then $Y_0 = \sum_1^{N_0} X_i$ is called a geometric convolution of $X$. Closely related is the random variable $Y = \sum_1^{N} X_i$ where $N = N_0+1$, which is also referred to as a geometric convolution.

Geometric convolutions arise naturally in many applied probability models. A recent paper of Gertsbakh (1984) discusses a rich variety of applications in reliability and queues and surveys research in the area, most of which was performed by Soviet authors. Feller (1971) Section XI.6 elegantly discusses terminating renewal processes, the time until termination being a geometric convolution. Several authors have studied random sampling or "thinning" of renewal processes which results in new renewal processes with geometric convolution interarrival times. Jacobs (1986) investigates a geometric convolution in the context of combining random loads and waiting for the stress to exceed a given level. In a $G|G|1$ queue in equilibrium the waiting time distribution is a geometric convolution and has been studied in this context by Szekli (1986) and Köllerström (1976). Finally, numerous applications arise in regenerative stochastic processes. Consider a regenerative process (Smith (1958)) where in each cycle an event A may or may not occur, independently of other cycles. The waiting time for $A$ to occur is then of the form:

$$W = \sum_1^{N_0} X_i + Z = Y_0 + Z \tag{1.1}$$

where $A$ occurs for the first time in cycle $N_0+1$, $N_0$ is geometricially distributed with parameter equal to the probability of $A$ occurring during a specified cycle, and $Z$ is the waiting time from the beginning of cycle $N_0+1$ until $A$ occurs.

Keilson (1966) recognized the prevalence of (1.1) and considered the case of small $p$. He showed that if the $X_i$ are non-negative with finite second moment then $W/EW$ converges in distribution to an exponential with mean $1$ as $p \to 0$. Thus the waiting time for a rare event (small p) to occur is approximately exponential. Solovyev, (1971) considered a sequence of random variables of the form (1.1) in which the distribution of $X$ varies with $p$ and obtained conditions for asymptotic exponentiality as $p \to 0$. Solovyev also obtained error bounds for the exponential approximation.

In this paper we seek to bound the distance between a geometric convolution of non-negative random variables and an exponential distribution with the same mean. This problem is cited by Gertsbakh (1984) as being "of great interest for engineering applications". Defining $X$, $Y$ and $Y_0$ as above, $q = 1-p$, $\mu = EX$, $\mu_2 = EX^2$, $\gamma = \mu_2/2\mu^2$, and $\overline{F}_{Y_0}(t) = \Pr(Y_0 > t)$ we derive:

$$(1.2) \qquad q e^{-\frac{p}{q}\left(\frac{t}{\mu} + 2\gamma - 1\right)} \le \overline{F}_{Y_0}(t) \le e^{-\frac{pt}{q\mu}} + \frac{\gamma p}{q} .$$

Defining $d(Y_0)$ as the sup norm distance between $Y_0$ and an exponential distribution with mean $EY_0 = q\mu/p$, it follows from (1.2) that for $0 < p \le 1/2$:

$$d(Y_0) \le 2\gamma p = \mu_2 p/\mu^2 . \tag{1.3}$$

The bound (1.3) is asymptotically (as $p \to 0$) sharp in the following sense. Since the problem is scale invariant, the true upper bound is a function of $\gamma$ and $p$, say $B(\gamma,p)$. We show (Section 4) that $2\gamma p/(2\gamma p+q) \le B(\gamma,p)$ and thus $B(\gamma,p) = 2\gamma p(1+o(1))$.

Bounds are also obtained for $d(Y_0+Z)$ where $Z$ is a non-negative random variable independent of $Y_0$, for $d(Y)$, and for $d(Y^*)$ where $Y^*$ is the stationary renewal distribution corresponding to $Y$ and $Y_0$.

The current bounds offer improvement over those of Solovyev (1971) in that they are derived under less restrictive conditions, require less information about $X$ and $Z$ to compute, and are in general tighter. This comparison is discussed in Section 4.

In the above $X$ is assumed non-negative with known first two finite moments. In Section 3 we restrict $X$ further and obtain improved bounds. Perhaps the most interesting of these results is that if $X$ is assumed NBUE (new better than used in expectation, defined in Section 2) then $d(Y_0)$ is exactly equal to $p$.

Our methodology is a combination of reliability and renewal theory geared to exploit the fact that $Y_0$ is NWU (new worse than used) as pointed out by Daley (1984) and Köllerström (1976). The technique of studying random variables through their aging properties was developed by several authors, most notably by Barlow, Marshall and Proschan, and is lucidly presented in the text of Barlow and Proschan (1975).

2. Bound for General X. A distribution $F$ on $[0,\infty)$ is defined to be NWU (new worse than used) if:

$$\overline{F}(t+x) \geq \overline{F}(t)\overline{F}(x) \quad \text{for all} \quad t, x \geq 0 . \tag{2.1}$$

Similarly NBU (new better than used) is defined by reversing the inequality in (2.1). Thus $F$ is NWU (NBU) if its survival distribution at age $0$ is stochastically smaller (larger) than its survival distribution at age $x$ for all $x > 0$.

Let $\{X_i, i \geq 0\}$ be an i.i.d. sequence of non-negative random variables, let $N_0$ be independent of this sequence with $\Pr(N_0=k) = q^k p$, $k = 0,1,\ldots$ and define $S_n = \sum_1^n X_i$, and $Y_0 = S_{N_0}$. The following simple but very useful result is due to Daley (1984) and Köllerström (1976).

Lemma 2.1. $Y_0$ is NWU.

Proof. For $t > 0$ let $Y_t$ be distributed as the condition distribution of $Y_0-t$ given $Y_0 > t$. Define $N_t = \max\{k:S_k \leq t\}$ and let $X_t$ be distributed as the conditional distribution of $S_{N_t+1}-t$ given $Y_0 > t$. Then:

$$Y_t \overset{st}{=} X_t + Y_0 \overset{st}{\geq} Y_0$$

where $X_t$ and $Y_0$ are independent. □

Lemma 2.2, below, is a slight generalization of a known result (Barlow and Proschan (1975), p. 162), the generalization allowing for an atom at zero which will be required for $Y_0$.

Lemma 2.2. Assume that $W$ is NWU with an atom of size $p$ at zero, but with no other atoms. Let $F$ be the cdf of $W$ and $M = \sum_0^\infty F^{(k)}$ the renewal function. Then:

$$\overline{F}(t) \geq qe^{-pq^{-1}} e^{-(M(t)-1)} \geq e^{-(M(t)-1)} .$$

Proof. Let $\{N_1(t)-N_1(0), t > 0\}$ be a non-homogeneous Poisson process with $E(N_1(t)-N_1(0)) = -Ln(\overline{F}(t)/q)$. This process has its first event epoch $T_1$ distributed as $W|W > 0$. its next interarrival time $T_2-T_1$ distributed as $W-T_1|W > T_1$ and in general its $k^{th}$ interarrival time $T_k-T_{k-1}$ distributed as $W-T_{k-1}|W > T_{k-1}$. Since W is NWU we have:

$$T_k-T_{k-1}|T_1 \cdots T_{k-1} \overset{st}{\geq} W \text{ for } k \geq 2 . \tag{2.2}$$

Next consider $\{N_2(t)-N_2(0), t \geq 0\}$ where $N_2$ is a renewal process with interarrival time $W$. The first event epoch of $N_2(t)-N_2(0)$ occurs at $S_1 \sim W|W > 0 \sim T_1$. Subsequent interarrival times are distributed as $W$. It follows from $S_1 \sim T_1$ and (2.2) that:

$$T_k \overset{st}{\geq} S_k \text{ for } k \geq 1 . \tag{2.3}$$

Thus from (2.3):

$$N_1(t)-N_1(0) \overset{st}{\le} N_2(t)-N_2(0) \quad (2.4)$$

Taking expectations in (2.4):

$$-\mathrm{Ln}(\overline{F}(t)/q) \le M(t)-q^{-1} . \quad (2.5)$$

Finally, the result follows from (2.5) and the observation that $qe^{-pq^{-1}} \ge q(1+pq^{-1}) = 1$. □

Corollary 2.1. Suppose that $X \ge 0$ has $F(0) < 1$ and $\mu_2 = EX^2 < \infty$. Then:

$$\overline{F}_{Y_0}(t) \ge qe^{+pq^{-1}(t\mu^{-1}+2\gamma-1)} = qe^{-pq^{-1}(2\gamma-1)}e^{-t/EY_0}$$

where $\gamma = \mu_2/2\mu^2$.

Proof. Case (i). We first consider the case in which $X$ has no atoms. Then $Y_0$ has an atom of size $p$ at zero and no other atoms. Define $G_p$ as a geometrically distributed random variable with parameter $p(\Pr(G_p=k) = q^k p,\ k=0,1,\ldots)$. Define $N_{Y_0}$ and $N_X$ as renewal processes with interarrival times $Y_0$ and $X$ respectively, and $M_{Y_0} = EN_{Y_0}$, $M_X = EN_X$.

Note that $N_{Y_0}$ has $1+G_p$ renewals at zero, and $G_p$ renewals at each renewal epoch of $N_X$ in $(0,\infty)$. Thus:

$$(2.6) \qquad M_{Y_0}(t) = q^{-1}+pq^{-1}(M_X(t)-1) = 1+pq^{-1}M_X(t) \ .$$

Alternatively (2.6) can be easily proved using Laplace transforms. Next, we note Lorden's (1970) upper bound for the renewal function:

$$(2.7) \qquad M_X(t) \leq \frac{t}{\mu} + 2\gamma \ .$$

The result now follows from Corollary 2.1, (2.6), (2.7) and $EY_0 = pq^{-1}\mu$.

<u>Case (ii)</u>. Now consider the general case. Define $e_n$ to be uniformly distributed on $(0,\varepsilon_n)$, $n = 1,2,\ldots$, with $\lim \varepsilon_n = 0$, $e_n$ independent of $X$, and $X_n = X+e_n$. Since $X_n$ converges to $X$ in quadratic mean, $\mu_n = EX_n \to \mu$ and $\gamma_n = EX_n^2/2\mu_n^2 \to \gamma$. Define $Y_{0,n} = \sum_1^{G_p} X_{n,i}$, the analogue of $Y_0$ with $X$ replaced by $X_n$. Then by choosing $X_{n,i} = X_i+e_{n,i}$ we have $E(Y_{0,n}-Y_0)^2 = E(\sum_1^{G_p} e_{n,i})^2 \to 0$ as $n \to \infty$. Thus $Y_{0,n}$ converges in quadratic mean and thus in distribution to $Y_0$. By Case (i):

$$(2.8) \qquad \overline{F}_{Y_{0,n}}(t) \geq qe^{-pq^{-1}(t\mu_n^{-1}+2\gamma_n-1)} \ .$$

It follows by letting $n \to \infty$ in (2.8) that the desired bound for $\overline{F}_{Y_0}(t)$ holds at all continuity points of $Y_0$. But since $\overline{F}_{Y_0}$ is right continuous and the bound is a continuous function of $t$, it follows that the inequality must hold for all $t \geq 0$. $\square$

A non-negative random variable $X$ with distribution $F$ is defined to be NWUE (new worse than used in expectation) if $\mu = EX < \infty$

and:

$$(2.9) \qquad E(X-t \mid X > t) \geq \mu \text{ for all } t \geq 0 .$$

Similarly NBUE (new better than used in expectation) has the inequality reversed in (2.9), which implies that $\mu$ must be finite. Define $X^*$ to be distributed as the stationary renewal distribution corresponding to $X$; $X^*$ has cdf $G(x) = \mu^{-1} \int_0^x \overline{F}(t)dt$. Let $h^*(x)$ to be the failure rate function of $X^*$ defined by $h^*(x) = \overline{F}(x)/\mu\overline{G}(x)$. Noting that $h^*(x) = [E(X-x \mid X > x)]^{-1}$ it follows that NWUE is equivalent to each of the following:

$$(2.10) \qquad h^*(x) \leq \mu^{-1} \text{ for all } x \geq 0$$

$$(2.11) \qquad X \overset{st}{\leq} X^* .$$

Moreover, (2.10) implies:

$$(2.12) \qquad X^* \overset{st}{\geq} \mu\varepsilon .$$

Similarly $X$ NBUE is equivalent to each of the reverse inequalities in (2.10) and (2.11), with (2.10) holding for all $x \geq 0$ with $\overline{F}(x) > 0$. Moreover NBUE distributions satisfy the reverse inequality in (2.12).

Define $c\varepsilon$ to be an exponentially distributed random variable with mean $c$. For $X_1 \sim F_1$, $X_2 \sim F_2$ define $\mathcal{D}(X_1,X_2) = \mathcal{D}(F_1,F_2) = \sup|F_1(t)-F_2(t)|$. Finally define $\rho_X = \left|\frac{\mu_2}{2\mu^2} - 1\right|$.

Lemma 2.3. If $X$ is either NWUE or NBUE then $\mathcal{D}(X^*,\mu\varepsilon) \leq \rho_X$.

Proof. Consider the NWUE case, the NBUE case being totally analogous. First, defining $A = \{t:\bar{F}(t) \geq e^{-t\mu^{-1}}\}$:

$$(2.13)\qquad \mathcal{D}(X^*,\mu\varepsilon) \leq \sup_B \left|G(B) - \int_B \mu^{-1}e^{-t\mu^{-1}}dt\right| = \mu^{-1}\int_A (\bar{F}(t)-e^{-t\mu^{-1}})dt .$$

From (2.11), (2.12) and (2.13):

$$(2.14)\qquad \mathcal{D}(X^*,\mu\varepsilon) \leq \mu^{-1}\int_A (\bar{G}(t)-e^{-t\mu^{-1}})dt \leq \mu^{-1}\int_0^\infty (\bar{G}(t)-e^{-t\mu^{-1}})dt = \rho_X. \quad \square$$

Define $Y^*$ to be the stationary renewal distribution corresponding to $Y_0$ and $Y$.

Corollary 2.2. If $X \geq 0$ with $F(0) < 1$ and $\mu_2 < \infty$, then $\mathcal{D}(Y^*,(EY_0)\varepsilon) \leq pq^{-1}\gamma$.

Proof. By simple computation $\rho_{Y_0} = pq^{-1}\gamma$. The result now follows from Lemmas 2.1 and 2.3. $\square$

Recall that $d(Y_0) = \mathcal{D}(Y_0,(EY_0)\varepsilon)$, the sup norm distance between $Y_0$ and an exponential distribution with the same mean.

Theorem 2.1. If $X \geq 0$ with $F(0) < 1$ and $\mu_2 < \infty$, then:

$$\text{(i)}\quad qe^{-pq^{-1}(2\gamma-1)}e^{-t/EY_0} \leq \bar{F}_{Y_0}(t) \leq e^{-t/EY_0} + \gamma pq^{-1}$$

$$\text{(ii)}\quad d(Y_0) \leq p\max(2\gamma,\gamma q^{-1}) = \begin{cases} 2\gamma p, & 0 < p \leq 1/2 \\ \gamma q^{-1}p, & p > 1/2 \end{cases}$$

Proof. The bound on the left of (i) is the conclusion of Corollary 2.1. The upper bound follows from Lemma 2.1, (2.11) and Corollary 2.2 by:

$$(2.15)\qquad \bar{F}_{Y_0}(t)-e^{-t/EY_0} \le \bar{F}_{Y^*}(t)-e^{-t/EY_0} \le \mathcal{D}(Y^*,(EY_0)\varepsilon) \le \gamma pq^{-1} .$$

Finally (ii) follows from (i) noting that:

$$(2.16)\qquad e^{-t/EY_0}-\bar{F}_{Y_0}(t) \le e^{-t/EY_0}[1-qe^{-pq^{-1}(2\gamma-1)}]$$

$$\le 1-q(1-pq^{-1}(2\gamma-1)) = 2\gamma p .$$

Thus, $d(Y_0) \le \max(2\gamma p,\gamma pq^{-1}) \le 2\gamma p$ for $p \le 1/2$. $\square$

Corollary 2.3. Under the conditions of Theorem 2.1, $d(Y) \le pq^{-1}\max(1+\gamma q^{-1},2\gamma-1) \sim p\max(1+\gamma,2\gamma-1)$.

Proof. Since $\bar{F}_Y(t) = q^{-1}\bar{F}_{Y_0}(t)$ we can multiply all three sides of inequality (i) of Theorem 2.1 obtaining an upper and lower bound for $\bar{F}_Y(t)$. Subtracting $e^{-t/EY}$ from both $F_Y(t)$ and the upper bound we obtain:

$$(2.17)\qquad \bar{F}_Y(t)-e^{-t/EY} \le \sup(q^{-1}e^{-t/EY_0}-e^{-t/EY}) + \gamma q^{-2}p$$

$$= pq^{-1}(1+\gamma q^{-1}) .$$

Subtracting $\bar{F}_Y(t)$ from $e^{-t/EY}$ and from the lower bound for $\bar{F}_Y(t)$ yields:

$$(2.18)\qquad e^{-t/EY}-\bar{F}_Y(t) \le \sup(e^{-t/EY}-be^{-t/EY_0})$$

where $b = e^{-pq^{-1}(2\gamma-1)}$.

For $b \geq q$, the right side of (2.18) equals $pq^{p/q}e^{2\gamma-1}$ which is bounded above by $p$, while for $b < q$ the right side of (2.18) equals $1-b$ which is bounded above by $pq^{-1}(2\gamma-1)$. Thus $d(y)$ is bounded above by the larger of $pq^{-1}(1+\gamma q^{-1})$ and $pq^{-1}(2\gamma-1)$, and the result is thus proved. □

We next seek bounds for $d(Y_0+Z)$ with $Z \geq 0$ and independent of $Y_0$. A few simple preliminary results are first presented.

First, for $c_1 < c_2$, a routine calculus argument proves:

$$(2.19) \qquad \mathcal{D}(c_1\varepsilon, c_2\varepsilon) = \left(1 - \frac{c_1}{c_2}\right)\left(\frac{c_1}{c_2}\right)^{c_1/c_2-c_1} \leq 1 - \frac{c_1}{c_2} .$$

Next, note that for any constant $\beta$:

$$(2.20) \qquad \mathcal{D}(W_1+\beta, W_2+\beta) = \mathcal{D}(W_1, W_2) .$$

It follows from (2.18) that for any random variable $V$ independent of $W_1$ and $W_2$ that:

$$(2.21) \qquad \mathcal{D}(W_1+V, W_2+V) \leq \mathcal{D}(W_1, W_2) .$$

Next let $Z \geq 0$ be independent of $\varepsilon$ with Laplace transform $\mathcal{L}$.

<u>Lemma 2.4</u>. $\mathcal{D}(c\varepsilon, c\varepsilon+Z) \leq 1-\mathcal{L}(c^{-1}) \leq 1-e^{-c^{-1}EZ} \leq c^{-1}EZ$.

<u>Proof</u>. Let $F(t) = 1-e^{-c^{-1}t}$. For $t \geq y \geq 0$:

$$(2.22) \qquad F(y) = (\overline{F}(t-y)-\overline{F}(t))/\overline{F}(t-y) \geq \overline{F}(t-y)-\overline{F}(t) .$$

For $0 \le t \le y$:

(2.23) $$F(y) = 1-\overline{F}(y) \ge 1-\overline{F}(t) = \overline{F}[(t-y)^{+}]-\overline{F}(t) .$$

Thus $F(y) \ge \overline{F}[(t-y)^{+}]-\overline{F}(t)$ for $y, t \ge 0$. Consequently:

(2.24) $$\Pr(c\varepsilon+Z > t)-\Pr(c\varepsilon > t) = E[\overline{F}(t-Z)^{+}-\overline{F}(t)] \le EF(Z) .$$

Now:

(2.25) $$EF(Z) = E[1-e^{-c^{-1}Z}] = 1-\mathcal{L}(c^{-1}) \le 1-e^{-c^{-1}EZ} . \quad \square$$

Finally, we need a simple but useful result:

Lemma 2.5. Suppose that $X$ and $Y$ are both either stochastically larger or stochastically smaller than Z. Then:

$$\mathcal{D}(X,Y) \le \max(\mathcal{D}(X,Z),\mathcal{D}(Y,Z)) .$$

Proof. Immediate.

The above inequalities now enable us to derive:

Theorem 2.2. Let $X$ be as in Theorem 2.1, and let $Z \ge 0$ be independent of $X$ with $EZ < \infty$. Define $\delta_Z = EZ/\mu$. Then:

(i) $d(Y_0+Z) \le [2\gamma+\delta_Z q^{-1}]p$, for $0 < p \le 1/2$.

(ii) $d(Y^{*}) \le \gamma p q^{-1}$ .

Proof. By the triangle inequality:

$$(2.26)\qquad d(Y_0+Z) \le \mathcal{D}(Y_0+Z,(EY_0)\varepsilon+Z)+\mathcal{D}((EY_0)\varepsilon+Z,(E(Y_0+Z))\varepsilon) .$$

By (2.20) and Theorem 2.1:

$$(2.27)\qquad \mathcal{D}(Y_0+Z,(EY_0)\varepsilon+Z) \le d(Y_0) \le 2\gamma p , \quad \text{for } 0 < p \le 1/2.$$

By Lemma 2.5:

$$(2.28)\qquad \mathcal{D}((EY_0)\varepsilon+Z,(E(Y_0+Z))\varepsilon) \le \max[\mathcal{D}((EY_0)\varepsilon+Z,(EY_0)\varepsilon)$$
$$\mathcal{D}((E(Y_0+Z))\varepsilon,(EY_0)\varepsilon)] .$$

By Lemma 2.4:

$$(2.29)\qquad \mathcal{D}((EY_0)\varepsilon+Z,(EY_0)\varepsilon) \le 1-e^{-EZ/EY_0} = 1-e^{-pq^{-1}\delta_Z} \le pq^{-1}\delta_Z .$$

By (2.19):

$$(2.30)\qquad \mathcal{D}((E(Y_0+Z))\varepsilon,(EY_0)\varepsilon) \le 1 - \frac{EY_0}{E(Y_0+Z)} \le \frac{EZ}{EY_0} = pq^{-1}\delta_Z .$$

Result (i) now follows from (2.27)-(2.30).

Finally by (2.12), Lemma 2.5, Corollary 2.2, and (2.19):

$$(2.31)\qquad d(Y^*) \le \max[\mathcal{D}(Y^*,(EY_0)\varepsilon),\mathcal{D}((EY^*)\varepsilon,(EY_0)\varepsilon)]$$
$$\le pq^{-1}\gamma .$$

Thus (ii) holds and the proof of Theorem 2.2 is complete.

3. Improved Bounds Under Additional Assumptions. In this section we outline the improvements in the results under various aging assumptions on the distribution of X.

3.1. NBUE. Suppose that X is NBUE distributed. Then $Y^* \overset{st}{=} Y_0 + X^* \overset{st}{\leq} Y_0 + X \overset{st}{=} Y$, thus Y is NBUE. Note that $Y^*$ is the stationary renewal distribution corresponding to both $Y_0$ and Y. It thus follows from (2.10) that:

$$\frac{p}{\mu} \leq h_{Y^*}^{(t)} \leq \frac{p}{q\mu} \quad \text{for all} \quad t \geq 0 . \tag{3.1.1}$$

Consequently:

$$e^{-\frac{px}{q\mu}} \leq \bar{F}_{Y^*}(t+x)/\bar{F}_{Y^*}^{(t)} \leq e^{-\frac{px}{\mu}} \quad \text{for all} \quad t,\ x \geq 0 . \tag{3.1.2}$$

Thus for p small, $Y^*$ has an approximate lack of memory in that the residual age distribution varies with t by at most p in sup norm.

Note that from (3.1.1), $(EY_0)\varepsilon \leq Y^* \leq (EY)\varepsilon$ and from (2.11), $Y_0 \overset{st}{\leq} Y^* \overset{st}{\leq} Y$. Since $\mathcal{D}(Y,Y_0) \leq p$ and $\mathcal{D}((EY_0)\varepsilon,(EY)\varepsilon) \leq p$ it follows that:

$$\max(d(Y^*,Y_0),d(Y^*,Y),d(Y^*,(EY_0)\varepsilon)) \leq p . \tag{3.1.3}$$

Furthermore by Lemma 2.3:

$$d(Y^*,(EY)\varepsilon) \leq \rho_Y = p\rho_X . \tag{3.1.4}$$

From (3.1.3) and (3.1.4), using the methodology of Section 2 it is straightforward to derive:

(3.1.5) $$d(Y_0) \leq p$$

(3.1.6) $$d(Y) \leq p$$

(3.1.7) $$d(Y^*) \leq p\rho_X$$

(3.1.8) $$d(Y_0+Z) \leq p(1+q^{-1}\delta_Z)$$

(3.1.9) $$\bar{F}_Y(t) \geq e^{-t/EY} - p\rho_X$$

(3.1.10) $$e^{-t/EY_0} \leq \bar{F}_Y(t) \leq q^{-1}e^{-t/EY} .$$

Note that $Pr(Y_0=0) = p/(1-qPr(X=0)) \geq p$. It follows that for any non-negative $X$ with finite mean:

(3.1.11) $$d(Y_0) \geq p .$$

Thus (3.1.5) and (3.1.11) show that for $F$ NBUE:

(3.1.12) $$d(Y_0) = p .$$

Finally, we mention that when $\rho_X < p/2$ we can improve on (3.1.6) by using Daley's (1986) bound for NBUE distributions applied to $Y$. This yields:

(3.1.13) $$d(Y) \leq \sqrt{2\rho_Y} = \sqrt{2p\rho_X}$$

3.2. NBU. If $X$ is NBU then an argument similar to the NBUE case of Section 3.1 shows that $Y$ is also NBU. Now $Y_0$ is NWU, $Y$ is NBU and $Y_0-t|Y_0>t \overset{st}{\geq} Y-t|Y>t$ for all $t \geq 0$. Thus:

(3.2.1) $$Y_0 \overset{st}{\leq} Y-t|Y>t \overset{st}{\leq} Y$$

(3.2.2) $$q\,\bar{F}_Y(x) \leq \frac{\bar{F}_Y(t+x)}{\bar{F}_Y(t)} \leq \bar{F}_Y(x) \quad \text{for all } x,\ t \geq 0 .$$

The residual age distributions thus cannot vary by more than $p$ in sup norm.

Since $Y$ is NBU we can derive an analogue of Corollary 2.1 for Y:

(3.2.3) $$\bar{F}_Y(t) \leq e^p e^{-t/EY} .$$

Combine (3.1.9), (3.1.10) and (3.2.3) to obtain

(3.2.4) $$\max(e^{-t/EY_0}, e^{-t/EY}-p\rho_X) \leq \bar{F}_Y(t) \leq e^p e^{-t/EY} .$$

3.3. NWUE. Assume that $X$ is NWUE distributed. It follows from the argument of Section 2.1 that $Y$ is also NWUE. Thus by Lemma 2.3:

(3.3.1) $$\mathcal{D}(Y^*, (EY)\varepsilon) \leq p\rho_X .$$

Since $Y \overset{st}{\leq} Y^*$ ((2.11)) it follows from (3.3.1) that:

(3.3.2) $$\bar{F}_Y(t)-e^{-t/EY} \leq \bar{F}_{Y^*}(t)-e^{-t/EY} \leq p\rho_X .$$

(3.5.3) $$d(Y_0+Z) \le p(\gamma+\delta_Z q^{-1}) .$$

<u>3.6</u>. <u>DFR</u>. A random variable on $[0,\infty)$ is defined to be DFR (decreasing failure rate) distributed if $X-t|X>t$ is stochastically decreasing in $t \ge 0$. Shantikumar (1986) recently proved that geometric convolutions of DFR are DFR. Thus if $X$ is DFR then so are $Y_0$, $Y$ and $Y^*$. Using the DFR property of $Y$ and $Y_0$ it follows from Brown (1983) p. 422 that:

(3.6.1) $$\max(d(Y),d(Y^*)) \le \frac{p\rho_X}{p\rho_X+1}$$

(3.6.2) $$d(Y_0) \le \frac{\rho_{Y_0}}{\rho_{Y_0}+1} = \frac{p\gamma}{p\gamma+q} .$$

A geometric convolution of DFR random variables arises naturally in the study of time to first failure for repairable systems (Brown (1984a) p. 611).

<u>3.7</u>. <u>IFR</u>. Assume that $X$ is IFR (increasing failure rate). Then it follows from Brown (1984b) that:

(3.7.1) $$M(t) \ge \frac{t}{\mu} + \frac{\sigma^2}{\mu^2}$$

where $M$ is the renewal function corresponding to $X$. Then (3.6.1) and an analogue of Corollary 2.1 yields:

(3.7.2) $$\bar{F}_Y(t) \le e^{-p(M(t)-1)} \le e^{2\rho_X p} e^{-t/EY} .$$

Since for $X$ NBUE, $0 \leq \rho_X \leq 1/2$, we see that (3.7.2) improves upon (3.2.3).

3.8. $\Pr(X=0) = \beta \varepsilon (0,1)$, Known. If $\beta = \Pr(X=0)$ is known, with $0 < \beta < 1$, then an improvement in the bound for $d(Y_0)$ can be achieved. Define $X'$ to be distributed as the conditional distribution of $X$ given $X > 0$. Then:

$$(3.8.1) \qquad \sum_1^{G_p} X_i \overset{st}{=} \sum_1^{G_{p^*}} X_i', \text{ where } p^* = p/(1-\beta q)$$

$$(3.8.2) \qquad \gamma_{X'} = (1-\beta)\gamma_X \ .$$

From Theorem (2.1):

$$(3.8.3) \qquad d(Y_0) \leq 2\gamma_{X'}p^* = (2\gamma_X p)(1-\beta)/(1-\beta q) \ .$$

The IMRL class is closed under the transformation $X' \to X$ for all $0 < \beta < 1$. Thus (3.5.1), (3.8.1) and (3.8.2) imply that for $X$ IMRL:

$$(3.8.4) \qquad d(Y_0) \leq (\gamma_X p)(1-\beta)/(1-\beta q) \ .$$

4. Comments and Additions.

4.1. Consider $X_\alpha \sim \mathrm{Bin}(1,\alpha)$, i.e. $\Pr(X_\alpha = 1) = \alpha$, $\Pr(X_\alpha = 0) = 1-\alpha$. Then $\gamma_\alpha = EX_\alpha^2/2(EX_\alpha)^2 = (2\alpha)^{-1}$, thus as $\alpha$ ranges from 1 to 0 $\gamma_\alpha$ ranges from $1/2$ to $\infty$. Thus all possible values of $\gamma$ are assumed by the $\mathrm{Bin}(1,\alpha)$ family. Now, let $Y_0(\alpha)$ be a sum of $G_p$, $X_\alpha$'s. Then:

$$\Pr(Y_0(\alpha)=0) = \frac{p}{p+q\alpha} = \left(\frac{2}{q+2\gamma p}\right)\gamma p \ . \tag{4.1.1}$$

It follows from (4.1.1) that if $B(\gamma,p)$ denotes $\sup d(Y_0)$, over all $Y_0$ with common $(\gamma,p)$ that:

$$B(\gamma,p) \geq \left(\frac{2}{q+2\gamma p}\right)\gamma p = 2\gamma p(1+o(1)) \ . \tag{4.1.2}$$

Thus as mentioned in the introduction the bound $2\gamma p$ for $d(Y_0)$ is asymptotically $(p \to 0)$ sharp.

4.2. Solovyev's (1971) bounds require existence and knowledge of $EX^m$ for some $m \in (2,3]$, and of $EZ^2$, in addition to our requirements. Defining $\gamma_m = [EX^m/(EX)^m]^{1/m-1}$, his bound are $O(\gamma_m p)$ as $p \to 0$ while ours are $O(\gamma p)$ (Solovyev's bounds depend on $\gamma_m/m-2$ and break down for $m = 2$). By the log convexity of moments (Marhsall and Olkin (1979) p. 74), $\gamma_m \geq \gamma$ for $m > 2$. The ratio $\gamma_m/\gamma$ can range from 1 to $\infty$ depending on $EX^m$. It is difficult to make a comparison to cover all possible cases but it appears that the current bounds are in general tighter. For example Solovyev's bound for $d(Y_0)$ corresponding to $m = 3$ is $6\gamma_3/\gamma \geq 6$ times as large as ours as $p \to 0$.

4.3. The bounds for $F_{Y_0}(t)$ derived in Sections 2 and 3 immediately yield bounds for the renewal function of a terminating renewal process (see Feller (1971) Section XI.6).

4.4. Defining $\lambda = 2\gamma p$, our upper bound for $d(Y_0)$ (Theorem 2.1) is $\lambda$. I conjecture that the best upper bound is $\lambda/\lambda+q$. This bound is achieved for the $\text{Bin}(1,\alpha)$ family discussed in Section 4.1.

4.5. Given a sequence $Y_{0,n} = \sum_1^{G(p_n)} X_{n,i}$, $n = 1,2,\ldots$ where $X_n$ and $p_n$ vary with $n$, it follows from Theorem 2.1 that $\lim \gamma_n p_n = 0$ is a sufficient condition for exponential convergence of $Y_{0,n}/EY_{0,n}$. This condition is not necessary as a slight modification of the example on p. 874 of Brown and Ge (1984) demonstrates. If the $X_n$ are all NBUE then it follows from p. 872 of Brown and Ge (1984) that a necessary and sufficient condition for exponential convergence of $Y_n/EY_n$ is $\lim p_n \rho_n = 0$ (recall $\rho_n = 1-\gamma_n$ in the NBUE case), and a necessary and sufficient condition for exponential convergence of $Y_{0,n}/EY_{0,n}$ is $\lim p_n = 0$.

4.6. A simple argument is now presented to show that under very general conditions geometric convolutions are asymptotically exponential as $p \to 0$.

Consider a random sequence $\{X_i, i \geq 1\}$ which obeys the strong law of large numbers for $\mu \varepsilon (0,\infty)$, that is:

$$\text{(4.6.1)} \qquad \Pr(\lim_{n\to\infty} \overline{X}_n = \mu) = 1$$

where $\overline{X}_n = \frac{1}{n} \sum_1^n X_i$. Define $G_p$ to a random variable which is geometrically distributed with parameter $p$. Consider $Y_0(p) = \sum_1^{G_p} X_i$. Now:

$$\text{(4.6.2)} \qquad pY_0(p) = (pG_p)\overline{X}_{G_p} .$$

It follows from (3.1.12) that $d(pG_p) = p$ and thus $pG_p$ converges in distribution to an exponential with mean 1. By (4.6.1), $\overline{X}_{G_p} \xrightarrow{a.s.} \mu$, thus $pY_0(p)$ converges in distribution to an exponential with mean $\mu$.

In the i.i.d. case $0 < EX < \infty$ suffices for exponential convergence of $Y_0$. It is not necessary that $X$ be non-negative, or that $G_p$ be independent of $\{X_i, i \geq 1\}$, or that $EX^2 < \infty$.

It is also seen that a large variety of dependent sequences lead to exponential convergence of geometric convolutions, for example stationary ergodic sequences with $0 < \mu < \infty$. An interesting problem is to obtain error bounds for $d(Y_0)$ (also $d(Y)$ and $d(Y_0+Z)$) for various classes of dependent sequences $\{X_i, i \geq 1\}$.

If we relax (4.6.1) to convergence in probability but impose the condition that $G_p$ be independent of $\{X_i\}$, then again $pY_0(p)$ is asymptotically exponential.

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